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Abstract:

In order to meet the objectives of workpackage WP1 a mathematical and geometrical model of the oculomotor plant able to implement the ocular eye movements in the 3D space has been developed. A detailed description of the eyeball and of the extra-ocular muscles is given. In particular the geometrical properties of the plant, necessary for the implementation of the Listing's Law, are described.
Two different models of the extra-ocular muscles are presented (Hill-type muscle and linearized muscle model). On the oculomotor plant the *Inverse* and *Direct Problem* are formulated and numerically solved in the 3D space. The first computes the muscle forces given a reference eye orientation, and the second one computes the eye orientation given the muscle forces.
These two control problems have been tested with a Simulator implementing the biomechanics of the oculomotor plant.

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1 Executive Summary

This document contains a detailed description of the oculomotor plant and a control strategy for the ocular movements in the 3D space, which constitutes the Deliverable D1.4a entitled *Control of voluntary transfer of fixations to new depth planes (Intermediate Version)* of the EU Project EYESHOTS. Deliverable D1.4 is part of the workpackage WP1: *Eye Movements for Exploration of the 3D Space*. In particular D1.4a is the first outcome of the activities of the worktasks **Task 1.3** *Control of voluntary eye movements in 3D*, and **Task 1.4** *Bioinspired Stereovision Robot System.*
The workpackage WP1 is devoted to the study of ocular mechanics and oculomotor control, for both single eye and conjugate movements. The target is to investigate how eye plant mechanics affects the strategies implemented by the brain to drive typical biological ocular motions (including saccades and smooth pursuit). A second goal is the study of the geometric and kinematic effects of ocular motions on image flows, for supporting the estimation of 3D information from ocular motions. Finally, from the engineering point of view the major expected achievement is to develop a bio-inspired stereoscopic robot system capable of emulating the ocular motions to be used during the planned experimental tests.
In order to meet these objectives it is necessary to develop a detailed model of the ocular mechanics and of the control strategy (at muscolar level) that realize the human eye movements in the 3D space.
A detailed mechanical model of the oculomotor system (i.e., eyeball and extraocular muscles) is presented, and the geometrical properties implementing the Listing's Law are described.
The control strategy solve the two main problems, common to both artificial and biological systems, called *Inverse and Direct Problem*.
This model has been also used to develop the stereo-vision robot, described in detail in the Deliverable 1.4: *Bioinspired Stereovision Robot System.*. With the Simulator the validity of the control strategy proposed has been verified.
The oculomotor model described in this document has been implemented in the MATLAB/Simulink envinronment as described in the Deliverable 1.4a, entitled: *Bioinspired Stereovision Robot System. Robot Prototype Simulator.*
The work presented can be easily integrated with the vision modules realized in the other workpachages (WP2, WP3), in order to realixe stereo vision experiments.
As final results these vision modules and the control strategy, presented here, can be used to control the robot-eye prototype developed on the model described in this document.

2 Introduction

This document is a major review of the human oculomotor system, composed of the eyeball and the extra-ocular muscles, and describes a new strategy to solve the two main control problems, called, *Direct and Inverse Problem*.
This study is the starting point to realize the control strategy for the human eye system and robot-eye prototype. In literature many different model of the oculomotor plant have been presented. In this document the mechanical and geometrical properties of the human eye are described in detail, and two different models of the extra-ocular muscles are presented.
The geometrical parameters of the oculomotor plant play a significant role for the implementation of ocular motions that obey to the so called *Listing's Law* and *Half-angle Rule*.
The extra-ocular muscles drive the eyeball, the eye movements are performed by the control strategy. In this document two main problems are analyzed and numerically solved: 1) Direct Problem, 2) Inverse Problem, which are common to both artificial and biological systems.
The Direct Problem computes the eye orientation given the muscle forces, instead the Inverse Problem computes the four muscle forces given a reference eye orientation. This control strategy is solved in the 3D space and respect the ocular laws.
Both the oculomotor model and the control strategy can be easily integrated with the vision module developed in the other workpackages (WP2, WP3). In fact given a retinal position (of the target object) the muscle forces that drive the eyeball in that position are computed.
Furthermore the control strategy presented here can be adapted to the robot-eye prototype, in fact the robot-eye is built on the assumption made in this document.
The validity of the control strategy has been tested with a simulator, developed with the MATLAB/Simulink envinronment described in the Deliverable 1.4a, instead the control algorithms have been developed in MATLAB code.
In the first part of this document is described in detail the oculomotor system (eyeball and muscles) and the two main ocular laws. In the second part the control strategy (Direct and Inverse problem) is presented and the simulations on the plant and on the control algorithms are shown.
Finally in the Appendix A is shown the MATLAB code implementing the two control algorithms.

3 Oculomotor Plant

In this Section the oculomotor plant system will be described, which is composed of the eyeball, the orbit, the pulleys and the extraocular muscles.
The human eye has an almost spherical shape with an average diameter between 23 mm and 26 mm, and weighs betwwen 7 g and 9 g. It is housed in a cavity in the skull called orbit and is protected by a mobile membrane (eyelid).
The human oculomotor system is composed of six extra-ocular muscles (EOMs), housed in the orbital cavity, which allow the eye to rotate about its center with negligible translations [1], [5]. This muscles give to the eye accelerations up to $20000\ degsec^{-2}$ allowing to reach angular velocities up to $900\ degsec^{-2}$ [3]. The

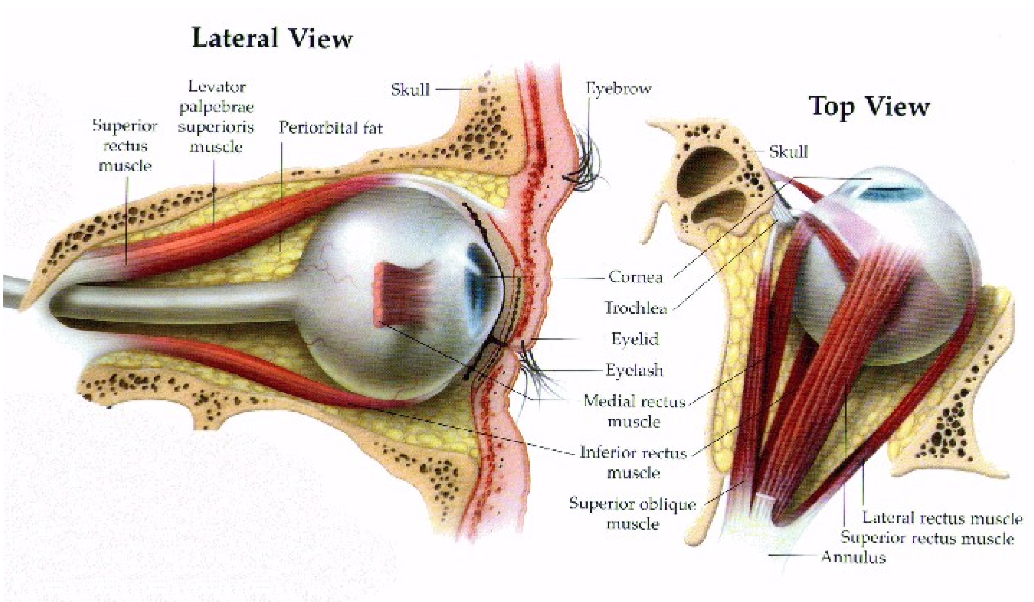


Figure 1: Lateral and top views of the human oculomotor plant

six extra-ocular muscles are coupled in three *agonistic/antagonistic* pairs, and classified in two groups: recti (*medial/lateral* and *superior/inferior*), and obliqui (*superior/inferior*). The four recti muscles have a common origin in the bottom of the orbit (*annulus of Zinn*); they diverge and run along the eye-ball up to their insertion points on the sclera (the eyeball surface). The insertion points form an angle of about 55 deg with respect to the optical axis and are placed simmetrically [1], [4]. The obliqui muscles have a more complex path within the orbit: they produce actions almost orthogonal to those generated by the recti, and are mainly responsible for the torsion of the eye about its optical axis.
Recent anatomical and physiological studies have suggested that the four recti have an important role for the implementation of saccadic motions which obey to

the so called *Listing's Law* and *Half-Angle rule*. In fact, it has been found that the path of the recti muscles within the orbit is constrained by soft connective tissue, named soft-pulleys [6], [7].
It has long been recognized that rectus EOMs of mammals contain two distinct layers: global and orbital layer [8], [9], [7] (Figure 2). The global layer is continuous from the annulus of Zinn to the tendinous insertion on the globe, the orbital layer terminates posterior on the soft-pulley tissue. On this caratteristics of the

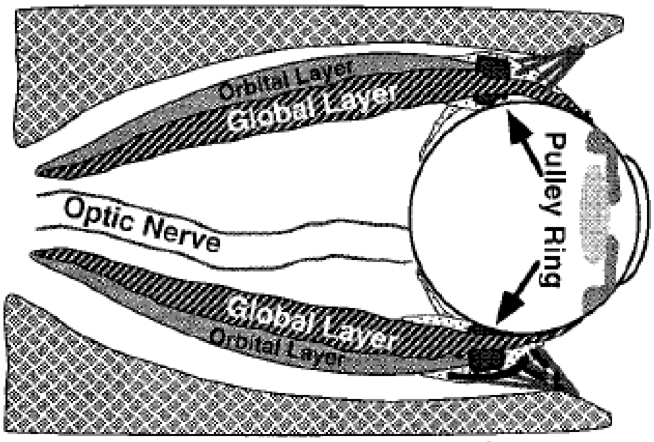


Figure 2: Global and orbital layers of the four recti extar-ocular muscles

muscle the active pulleys theory has been proposed, against the previous passive pulley theory, where both describe the movements of the pulley within the orbit [6], [7], [10], [11]. In the second theory is supposed that EOMs slide freely throught connective tissue sleves, which are elastically stabilized relative to the orbital wall.
On the contary the active pulley theory supposes that the EOMs, inserted in their pulley sleeves, move them longitudinally (anteriorly and posteriorly), but resisting to transverse movement.

3.1 Listing's Law and Half-Angle Rule

The main goal of this Section is to provide the mathematical formulation of Listing's law and of the Half-Angle Rule.
Recent anatomical advances [12],[7], suggest that the mechanics of the eye plant could play a significant role to implement Listing's law [13],[4],[14],[15],[16]. In fact, there is evidence that connective tissue within the orbit (referred in the literature as soft pulleys) constrains the path of the extraocular muscles, affecting the characteristics of ocular motions. For ocular motions, Listing's law defines the

amount of torsion of the eye for each direction of fixation [17],[18].
Relevant eye movements such as saccades and smooth pursuit obey to the Listing's law, and in the following paragraphs, we shall refer to Listing compatible motions to define any ocular movement respecting it.
Listing's Law states that there exists a specific orientation of the eye (with respect to a head fixed reference frame $< h >= \{h1, h2, h3\}$), called primary position. During saccades any physiological orientation of the eye (described by the frame $< e >= \{e1, e2, e3\}$), with respect to the primary position, can be expressed by a rotation vector, $\boldsymbol{v}$, always belongs to a head fixed plane, $\mathscr{L}$ (called Listing's plane). The normal to plane $\mathscr{L}$ is the eye direction of fixation at the primary position. Without loss of generality we can assume that ***e3*** is the fixation axis of the eye, and that $< h >\equiv< e >$ at the primary position: then, $\mathscr{L} = span\{h1, h2\}$, or equivalently we can state that Listing's plane is orthogonal to vector ***h3***. Figure 3 shows the geometry of Listing compatible rotations. In order to ensure that

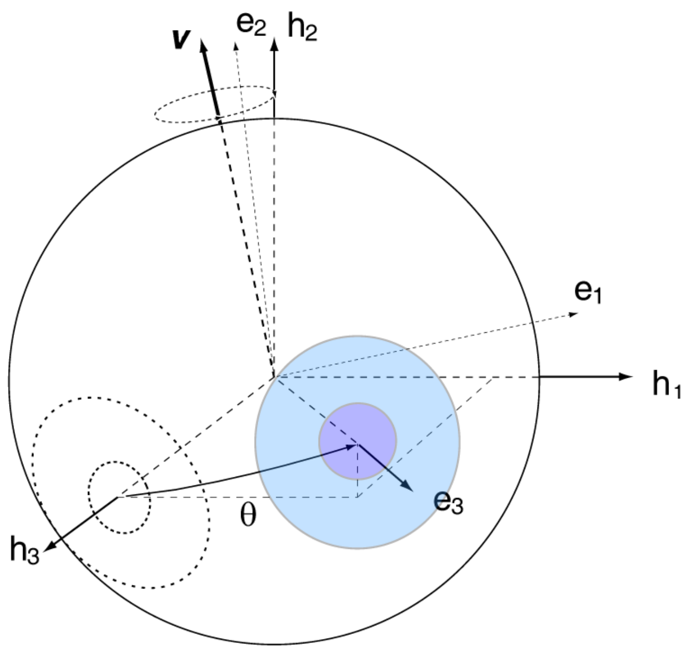


Figure 3: Geometry of Listing compatible rotations. The finite rotation axis $\boldsymbol{v}$ is always orthogonal to ***h3***. θ is the amount of rotation about $\boldsymbol{v}$

$\boldsymbol{v} \in \mathscr{L}$ at any time, the eye's angular velocity $\boldsymbol{\omega}$, must belong to a plane $\mathscr{P}_\omega$, passing through $\boldsymbol{v}$, whose normal, $\boldsymbol{n}_\omega$, forms an angle of $\theta/2$ with the direction of fixation at the primary position (see Figure 4). This property, directly implied by Listing's Law, is usually called Half-Angle Rule, [18],[19]. During a generic saccade the plane $\mathscr{P}_\omega$ is rotating with respect to both the head and the eye due to its dependency from $\boldsymbol{v}$ and θ. The normal to plane $\mathscr{P}_\omega$ is:

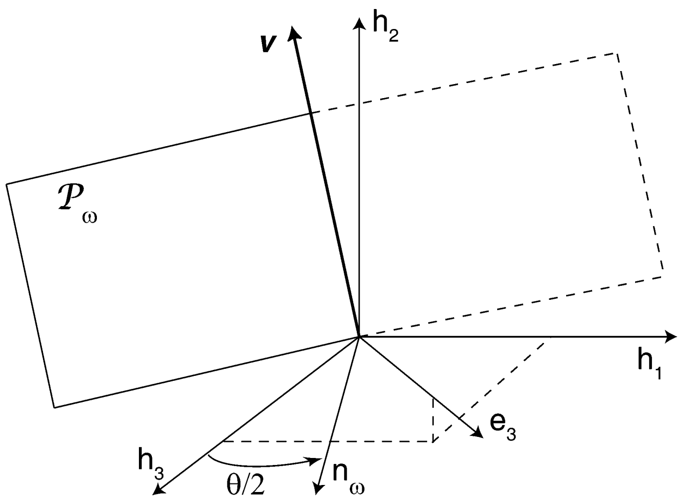


Figure 4: Half-angle rule geometry. The eye's angular velocity must belong to the plane $\mathscr{P}_\omega$ passing through axis $\boldsymbol{v}$ (dashed lines indicate the part of $\mathscr{P}_\omega$ behind Listing's plane). Vector $\boldsymbol{n_\omega}$ is orthogonal to $\mathscr{P}_\omega$.

$$\boldsymbol{n_\omega} = \boldsymbol{h_3} \cos\frac{\theta}{2} - (\boldsymbol{h_3} \times \boldsymbol{v}) \sin\frac{\theta}{2} \tag{1}$$

so that the half angle rule can be expressed as:

$$\boldsymbol{n_\omega} \cdot \boldsymbol{\omega} = 0 \tag{2}$$

3.2 Eye Model

In the previous Section we have seen that the human eye has an almost spherical shape and it is actuated by six extra-ocular muscles (EOMs). The eye-ball (globe) is modeled as a homogeneus sphere of radius r, with three degrees of freedom about its center.

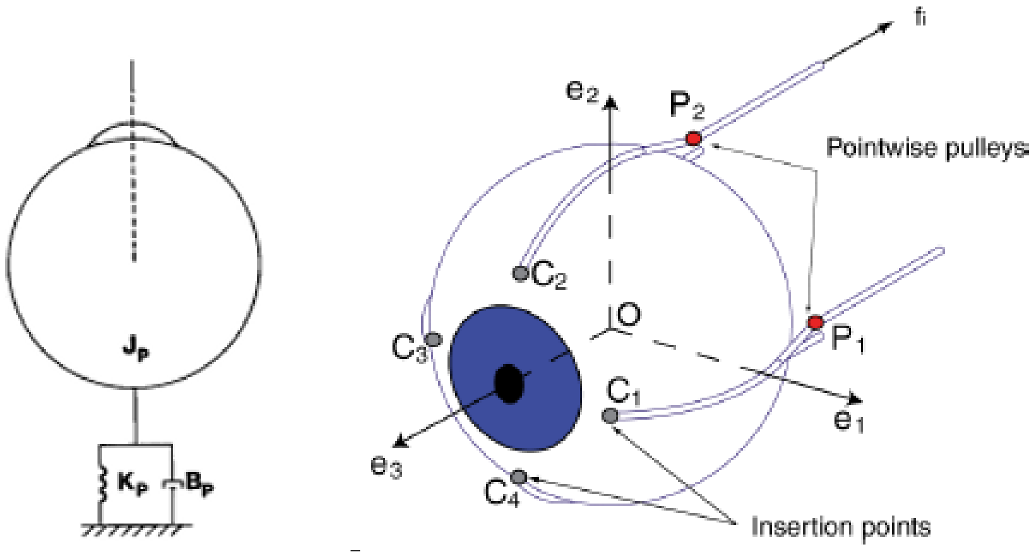
The globe has a moment of inertia J_p, connected to a viscous element B_p and a passive elasticity K_p modeling the visco-elastic properties of the orbital tissue (Figure 5a)[21],[22],[23],[24]. On this mechanical assumptions the total torque $\boldsymbol{\tau}$ that acts on the globe can be written as:

$$\boldsymbol{\tau} = J\dot{\boldsymbol{\omega}} + B\boldsymbol{\omega} + K\int_0^t \boldsymbol{\omega} dt \tag{3}$$

Where $\dot{\boldsymbol{\omega}}$, $\boldsymbol{\omega}$ are respectively the angular acceleration and velocity of the eye-ball, and $\frac{d}{dt}\boldsymbol{\omega}$=$\boldsymbol{\theta}$ is the angular position of the globe. J, B, K are the inertia, viscosity and elasticity matricies:

$$J = \begin{bmatrix} Jp & 0 & 0 \\ 0 & Jp & 0 \\ 0 & 0 & Jp \end{bmatrix} B = \begin{bmatrix} Bp & 0 & 0 \\ 0 & Bp & 0 \\ 0 & 0 & Bp \end{bmatrix} K = \begin{bmatrix} Kp & 0 & 0 \\ 0 & Kp & 0 \\ 0 & 0 & Kp \end{bmatrix}$$

As it appears in Figure 5b the four extra-ocular muscles are connected to the eyeball throught the insertion points and routed throught head fixed pointwise pulleys, emulating the soft-pulley tissue. The pointwise pulleys are located on the rear of the eyeball and their position can be described by vector $\mathbf{p_i} = P_i - O$, whereas the insertion points can be described by vectors $\mathbf{c_i} = C_i - O$ when the eye-ball is in primary position (Figure 5b) [18].



(a) (b)

Figure 5: Mechanical and geometrical model of the oculomotor plant. (a) Sketch with the mechanical parameters of the human eye: J_p moment of inertia, B_p orbit visocsity and K_p orbit elasticity. (b) Sketch of the eye at the primary position, according to the assumptions made on the placement of IPs, PPs, and routing of the EOMs. Each EOM generates a pulling force f_i.

When the eye is rotated about a generic axis $\boldsymbol{v}$ by an angle θ, the position of the insertion points can be expressed as:

$$\boldsymbol{r_i} = R(\boldsymbol{v}, \theta)\boldsymbol{c_i} \qquad \forall i = 1 \ldots 4 \tag{4}$$

Each extra-ocular muscle is assumed to follow the shortest path from each insertion point to the corresponding pulley, [10]. Therefore, the direction of the torque applied to the eye by the pulling action of each extra-ocular muscle is ortogonal to the plane defined by vectors $\mathbf{r_i}$ and $\mathbf{p_i}$ and can be expressed by the following formula:

$$\boldsymbol{m_i} = \frac{\boldsymbol{r_i} \times \boldsymbol{p_i}}{|\boldsymbol{r_i} \times \boldsymbol{p_i}|} \qquad \forall i = 1 \dots 4 \tag{5}$$

According to the given assumptions, the torque applied to the eye by the action of each EOM is given by:

$$\boldsymbol{\tau_i} = \tau_i \boldsymbol{m_i} \qquad \forall i = 1 \dots 4 \tag{6}$$

where $\tau_i = Rf_i \geqslant 0$, and f_i is the magnitude of the pulling forces generated by the ith EOM. Now it is important to show that, for any eye orientation compatible with the Listing's Law, all the torque directions $\boldsymbol{m_i}$ produced by the four recti extra-ocular muscles belong to a common plane (Halfe-Angle Plane $\mathscr{P}_\omega$) passing through the finite rotation axis $\boldsymbol{v}$ that lies on the Listing plane [18]. On these assumption all the vectors $\boldsymbol{m_i}$ are ortogonal to the vector $\boldsymbol{n_\omega}$, normal to the Half-Angle Plane, and we can rewrite equation (3) as:

$$\boldsymbol{m_i} = \frac{\boldsymbol{n_\omega} \times \boldsymbol{r_i}}{|\boldsymbol{n_\omega} \times \boldsymbol{r_i}|} \qquad \forall i = 1 \dots 4 \tag{7}$$

Therefore, the total torque generated by the action of the recti EOMs can be rewritten as:

$$\boldsymbol{\tau} = \boldsymbol{n_\omega} \times \sum_{i=1}^{4} \gamma_i \boldsymbol{r_i} \tag{8}$$

where:

$$\gamma_i = \frac{\tau_i}{|\boldsymbol{n_\omega} \times \boldsymbol{r_i}|} \geqslant 0 \qquad \forall i = 1 \dots 4 \tag{9}$$

At this point it is possible to rewrite the dynamic model of the oculomotor plant shown in equation (3) as:

$$J\dot{\boldsymbol{\omega}} + B\boldsymbol{\omega} + K \int_0^t \boldsymbol{\omega} dt = \boldsymbol{n_\omega} \times \sum_{i=1}^{4} \gamma_i \boldsymbol{r_i} \tag{10}$$

3.3 Hill Muscle Model

As we have seen in the previous Section the torque $\boldsymbol{\tau_i}$ is generated from the four rectus muscle. The EOMs are modeled according to Hill's approach [26],[25],[23]. In Figure 6 is shown a sketch of the Hill type model of a extraocular muscle.
The tendon of length l_m is in series and off-axis by a pennation angle α with the tendon of lenght l_t, the total length of the extra-ocular muscle is l_{tm}. The muscle is composed of two main components: the active force generator in parallel to

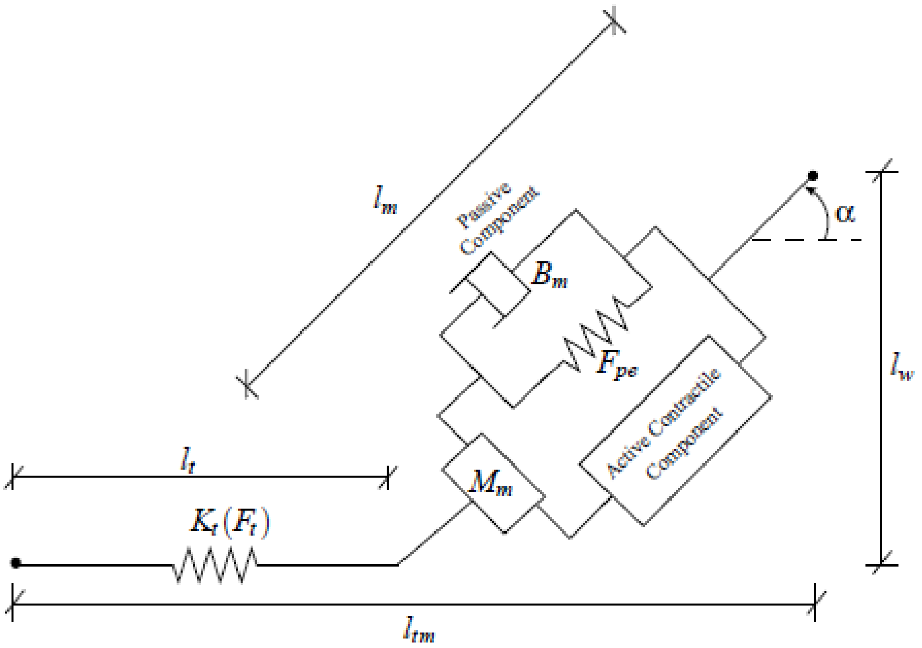


Figure 6: Hill-type model of the musculotendon dynamic.

the passive components. These two components are in series to the tendon elastic element K_t. The passive components is composed of the parallel combination of an elastic element F_{pe}, which describes the passive muscle elasticity, and a damping element B_m, which corresponds to the passive muscle viscosity. The active state generator generates the active force for the muscle, which is the product of length-tension relation $f_l(l_m)$, velocity tension relation $f_v(\dot{l}_m)$, and the activation level $a(t)$.
The mass of the muscle can be ignored and the total force generated by the extraocular muscle can be written as:

$$F_t = F_{act} + F_{pe} + B_m \dot{l}_m \tag{11}$$

where F_t, F_{act}, F_{pe} are, respectively, the tendon force, the active and the passive force in the muscle whereas $B_m \dot{l}_m$ is the viscous passive muscle force. The total torque for each muscle can be written as:

$$\boldsymbol{\tau_i} = \boldsymbol{F_{ti}} \times \boldsymbol{r_i} \qquad \forall i = 1 \dots 4 \tag{12}$$

where $\boldsymbol{r_i}$ is the vector which identify the position of the insertion point of the ith muscle.

3.4 Linearized Muscle Model

In order to simplify the oculomotor plant model several researchers have been developed a linearized model of the skeletal muscle. In particular we analyze the model introduced by Bahill [21], and used afterwards by Enderle [22] as a starting point for the development of control for saccadic and smooth pursuit eye movements.
The four rectus EOMs form two agonist-antagonist muscle pairs. The agonist muscle, for each muscle pair, is modelled as a parallel combination of an active state tension generator F_{AG}, viscosity element B_{AG}, and elastic element K_{LT}, connected to a series elastic element K_{SE}. Similarly the antagonist muscle is modeled as a parallel combination of an active state tension generator F_{ANT} , viscosity element B_{ANT} , and elastic element K_{LT} , connected to a series elastic element K_{SE}. Each of the elements defined in the model of the muscles is ideal and linear.

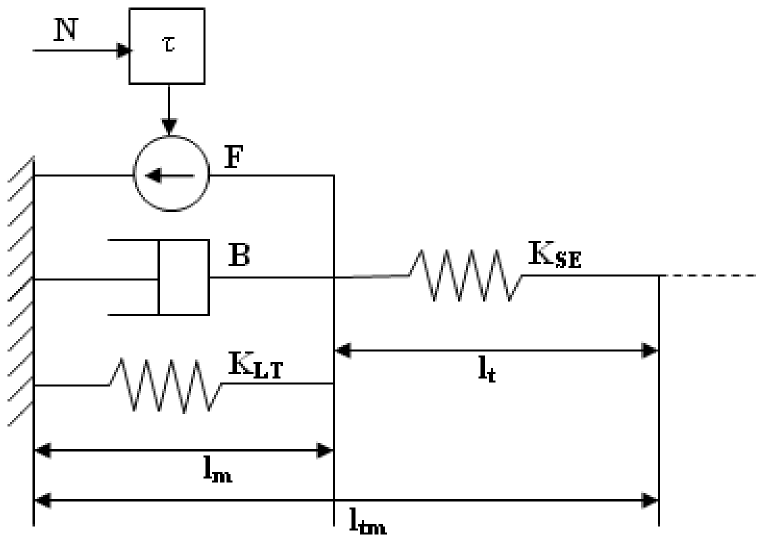


Figure 7: Linearized muscle model: F active state tension generator, B viscosity element, K_{LT} tendon elastic element, K_{SE} muscle elastic element.

In Figure 7 is shown a sketch of the linear model of a extra-ocular muscle, where B can be B_{AG} or B_{ANT} if the muscle is agonist or antagonist. The force generated by each mucle is:

$$F_t = K_{SE} l_{tm} \qquad F_t = F - K_{LT} l_t - B\dot{l}_m \tag{13}$$

where F is the active force, $K_{LT} l_t$ and $B\dot{l}_m$ are respectively the passive elastic and passive viscous force. The total torque generated by each muscle can be written as:

$$\boldsymbol{\tau_i} = \boldsymbol{F_{ti}} \times \boldsymbol{r_i} \qquad \forall i = 1 \ldots 4 \tag{14}$$

The active state tension generator F is a first-order filter of the neuronal control signal N, thus we have:

$$\dot{F} = \frac{N - F}{\tau} \tag{15}$$

where τ is the activation or deactivation time constant.

4 Control Problem

In the previous Section the eyeball has been modeled like a sphere with three degrees of freedom about its center and it is controlled by six extra-ocular muscles. During the discussion of the control strategy proposed we consider only the four recti muscles (lateral/medial and superior/inferior) which play a significant role during saccadic and smooth pursuit movements.
The control problem can be divided in two main parts: 1) computing the EOM forces to achieve a given (Listing compatible) eye orientation, this is the so called *Static Inverse Problem*; 2) computing eye orientation from EOM forces, called the *Static Direct Problem*. In particular the inverse problem is to compute the neurological control signal (muscle acivation) that implements the muscle force for a reference eye position, thus we have a mapping between the angular position of the eyeball and the motor commands.

4.1 Static Direct Problem

The direct dynamic problem is common to both biological and artificial motor system and a number of analytical solutions have been proposed. We can define the direct dynamic problem as: ”*given the force and the torque of the joints computing the position of the end effector*”.
In our case we have the eyeball (end-effector) with three degrees of freedom about its center (spherical joint) actuated by the four rectus muscles. Therefore, we can rewrite the direct dynamics problem for the oculomotor plant as: ”*given the four muscle forces computing the angular position of the eye*”. We want to solve this problem at steady state condition, which means computing the eye orientation given the EOM forces f_i^* without considering the transient of the oculomotor system. For the static direct problem the solution is unique, namely, four muscles force identify one, and only one, eye orientation. As it appears in the previous Section the dynamics model of the oculomotor plant, at steady state, is:

$$K\boldsymbol{v}\theta = \boldsymbol{n_\omega^R} \times \sum_{i=1}^{4} \gamma_i^* \boldsymbol{c_i} \qquad \gamma_i^* = \frac{f_i^* r}{|\boldsymbol{n_\omega^R} \times \boldsymbol{c_i}|} \geqslant 0 \qquad \forall i = 1 \dots 4 \tag{16}$$

For the static direct problem there is not an analitycal or geometrical solution for the 3D model presented in this document. We have found a recursive algorithm, which in few steps compute the eye orientation from the four muscle forces. Mathematically the algorithm can be written as:

$$\boldsymbol{\gamma_i}[K] = D[K]\boldsymbol{f^*}[K]r \qquad \forall K = 1 \dots N \tag{17}$$

where:

$$D[K] = \begin{bmatrix} \frac{1}{|\boldsymbol{n_\omega^R}[K-1]\times \boldsymbol{c_1}|} & 0 & 0 & 0 \\ 0 & \frac{1}{|\boldsymbol{n_\omega^R}[K-1]\times \boldsymbol{c_2}|} & 0 & 0 \\ 0 & 0 & \frac{1}{|\boldsymbol{n_\omega^R}[K-1]\times \boldsymbol{c_3}|} & 0 \\ 0 & 0 & 0 & \frac{1}{|\boldsymbol{n_\omega^R}[K-1]\times \boldsymbol{c_4}|} \end{bmatrix} \quad \boldsymbol{f}^* = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}$$

r is the radius of the eyeball and K represents the K-th iteration. The axis of rotaion $\boldsymbol{v}$ at the K-th iteration is:

$$\boldsymbol{v}[K] = \frac{\boldsymbol{n_\omega^R}[K-1] \times \sum_{i=1}^{4} \gamma_i[K]\boldsymbol{c_i}}{|\boldsymbol{n_\omega^R}[K-1] \times \sum_{i=1}^{4} \gamma_i[K]\boldsymbol{c_i}|} \qquad \forall i = 1\dots 4, \qquad \forall K = 1\dots N \tag{18}$$

and the angle θ of rotation about the axis $\boldsymbol{v}$ at the K-th iteration is:

$$\theta[K] = \frac{|\boldsymbol{n_\omega^R}[K-1] \times \sum_{i=1}^{4} \gamma_i[K]\boldsymbol{c_i}|}{K_p} \qquad \forall i = 1\dots 4, \qquad \forall K = 1\dots N \tag{19}$$

Finally the vector orthogonal to the Half-angle Plane can be written as:

$$\boldsymbol{n_\omega^R}[K] = \boldsymbol{h_3}\cos\frac{\theta[K]}{2} - (\boldsymbol{h_3} \times \boldsymbol{v}[\boldsymbol{K}])\sin\frac{\theta[K]}{2} \qquad \forall K = 1\dots N \tag{20}$$

where the initial value for the vector $\boldsymbol{n_\omega^R}$ is:

$$\boldsymbol{n_\omega^R}[0] = \boldsymbol{h_3} \tag{21}$$

4.2 Static Inverse Problem

The three rotational degrees of freedom of the eye are controlled by the four extraocular muscles, so there are infinitely values of the motor commands that correspond to a unique eye position. For example, in static condition, the tensions of the muscles can be encreased, so that the total force dose not change, leaving the position unchanged.

In the oculomotor system the four rectus muscles are divided in two agonistic/antagonistic muscle pairs (lateral/medial and superior/inferior) and are reciprocally innervated. This means that an increase in the innervation in the agonistic muscle is accompanied by a decrease in the antagonistic muscle innervation.

Here we investigate the problem of associating the motor commands that mantain the eyeball in a given reference eye position. All the mechanical and geometrical properties described in Section 4 are considered to solve the Inverse Problem, where we have defined the mathematical model of the oculomotor dynamic as:

$$\boldsymbol{\tau} = J\dot{\boldsymbol{\omega}} + B\boldsymbol{\omega} + K\boldsymbol{\theta} \tag{22}$$

where:

$$\boldsymbol{\tau} = \boldsymbol{n_\omega} \times \sum_{i=1}^{4} \gamma_i \boldsymbol{r_i}, \qquad \gamma_i = \frac{f_i r}{|\boldsymbol{n_\omega} \times \boldsymbol{r_i}|} \geqslant 0 \qquad \forall i = 1 \ldots 4 \tag{23}$$

The problem is to compute the forces f_i that mantain the eye (at steady state) in a given Listing compatible orientation. At steady state the angular velocity ($\boldsymbol{n_\omega}$) and the angular acceleration ($\dot{\boldsymbol{\omega}}$) of the eyeball are equal to zero, and let ($\boldsymbol{v}^*$,θ^*) be the reference orientation of the eye, the dynamic of the oculomotor plant can be re-written as:

$$K\boldsymbol{v}^*\theta^* = \boldsymbol{n_\omega} \times \sum_{i=1}^{4} \gamma_i \boldsymbol{r_i} \qquad \gamma_i = \frac{f_i r}{|\boldsymbol{n_\omega} \times \boldsymbol{r_i}|} \geqslant 0 \qquad \forall i = 1 \ldots 4 \tag{24}$$

where $\boldsymbol{n_\omega}$ and $\boldsymbol{r_i}$ are explicit function of ($\boldsymbol{v}^*$,θ^*):

$$\boldsymbol{r_i} = R(\boldsymbol{v}^*, \theta^*)\boldsymbol{c_i} \qquad \forall i = 1 \ldots 4, \qquad \boldsymbol{n_\omega} = \boldsymbol{h_3} \cos\frac{\theta^*}{2} - (\boldsymbol{h_3} \times \boldsymbol{v}^*) \sin\frac{\theta^*}{2} \tag{25}$$

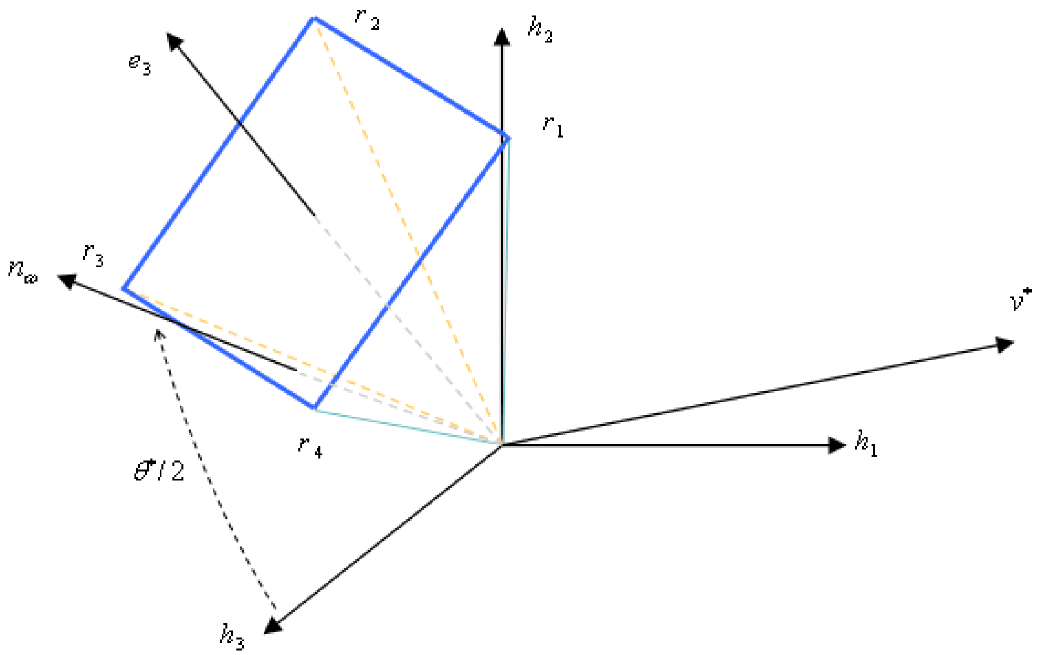


Figure 8: Geometry of the oculomotor plant after a rotation ($\boldsymbol{v}^*$,θ^*): $\boldsymbol{n_\omega}$ vector $\perp$ to the Half-Angle Plane, $\boldsymbol{r_i}$ new position of the four insertion points.

In Figure 8 are shown the position of the inserion points and of the vector orthogonal to the Hal-Angle Plane after a rotation around the axis $\boldsymbol{v}^*$ by an angle θ^*.

Applying a rigid rotation ($\boldsymbol{v}^*$,$-\theta^*$) the steady state equations can be re-written as:

$$K\boldsymbol{v}^*\theta^* = \boldsymbol{n}_{\boldsymbol{\omega}}^{\boldsymbol{R}} \times \sum_{i=1}^{4} \gamma_i \boldsymbol{c_i} \qquad \gamma_i = \frac{f_i r}{|\boldsymbol{n}_{\boldsymbol{\omega}}^{\boldsymbol{R}} \times \boldsymbol{c_i}|} \geqslant 0 \qquad \forall i = 1 \ldots 4 \tag{26}$$

In Figure 9 is shown the geometry of the oculomotor plant (insertion points and

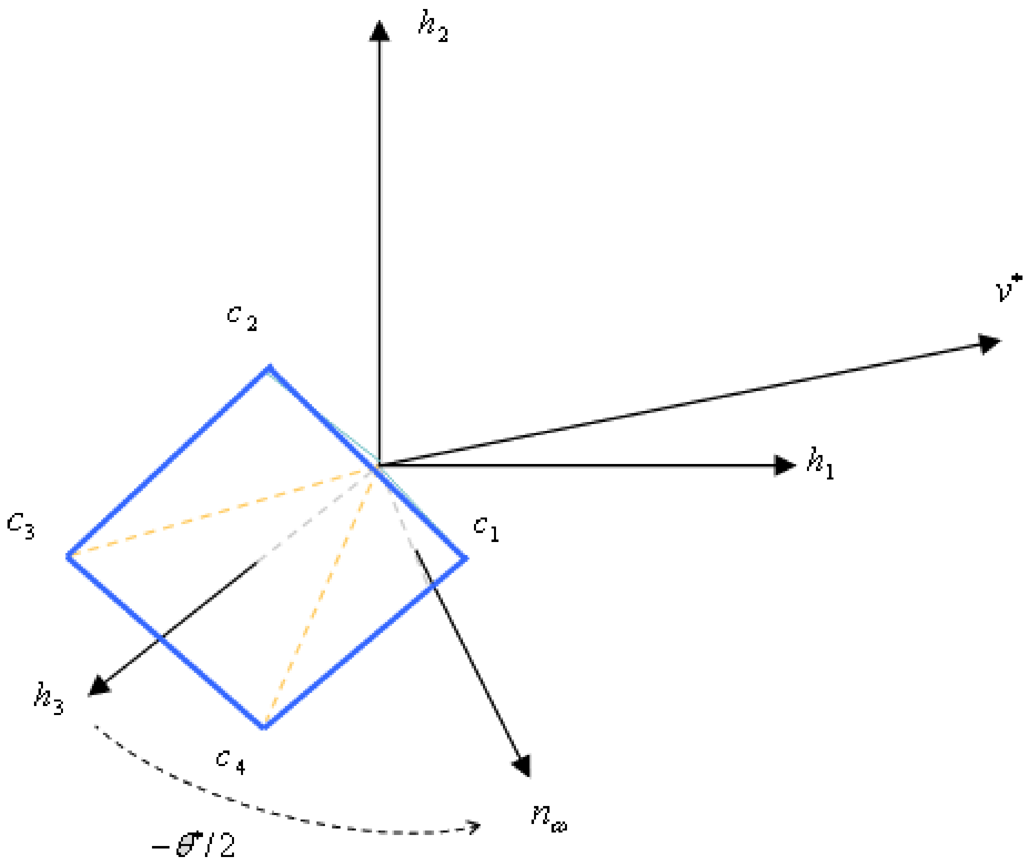


Figure 9: Geometry of the oculomotor plant after a rotation ($\boldsymbol{v}^*$,$-\theta^*$): $\boldsymbol{n}_{\boldsymbol{\omega}}^{\boldsymbol{r}}$ vector $\perp$ to the Half-Angle Plane, the vectors $\boldsymbol{r_i}$ and vector $\boldsymbol{e_3}$ are rotated back in primary position.

$\boldsymbol{n}_{\boldsymbol{\omega}}^{\boldsymbol{R}}$) after the rotation ($\boldsymbol{v}^*$,$-\theta^*$).
At this point the unknown variables are the γ_i and from these it is possible compute the four muscle forces.
With this approach we have transformed the problem of computing the muscle forces from a three dimensional space in to a bidimensional one, that is the plane identified by vectors $\boldsymbol{h_1}, \boldsymbol{h_2}$ or the locally isomorph surface.
The unknown variables can be expressed as:

$$\gamma_i = \gamma_i^p + \gamma_i^o \tag{27}$$

where γ_i, the general solution to the equation (22), is the sum of the general solution (γ_i^o) of the related homogeneous equation and the particular solution (γ_i^p). The particular solution is computed by assuming that the four γ_i^p are equal:

$$\gamma_i^p = \gamma^p \geqslant 0 \qquad \forall i = 1 \dots 4 \tag{28}$$

We can rewrite equation (22) as:

$$K\boldsymbol{v}^*\theta^* = \left\{ \boldsymbol{n_\omega^R} \times \sum_{i=1}^{4} \boldsymbol{c_i} \right\} \gamma^p \tag{29}$$

And the solution γ^p is:

$$\gamma^p = \frac{K\theta}{\sin\left(\frac{\theta}{2}\right) 4\cos\beta} \tag{30}$$

At this point the muscle forces related to the particular solution are:

$$f_i^p = \frac{\gamma^p |\boldsymbol{n_\omega^R} \times \boldsymbol{c_i}|}{r} \tag{31}$$

where r is the radius of the eyeball.
The γ_i^o are the general solution of the related homogeneous equation, which can be written as:

$$0 = \boldsymbol{n_\omega^R} \times \sum_{i=1}^{4} \gamma_i^p \boldsymbol{c_i} \qquad \forall i = 1 \dots 4 \tag{32}$$

This equation is true if:

$$\sum_{i=1}^{4} \gamma_i^p \boldsymbol{c_i} = \boldsymbol{n_\omega^R} \qquad \forall i = 1 \dots 4 \tag{33}$$

It is possible to redefine $\boldsymbol{n_\omega^R}$ as:

$$\tilde{\boldsymbol{n}}_{\boldsymbol{\omega}}^{\boldsymbol{R}} = \delta \boldsymbol{n_\omega^R} = [\tilde{n}_{\omega x}^R \; \tilde{n}_{\omega y}^R \; \tilde{n}_{\omega z}^R]^T \qquad \text{where} \qquad \delta = \frac{\sin\frac{\theta}{2}}{\cos\beta} \tag{34}$$

Where $\tilde{\boldsymbol{n}}_{\boldsymbol{\omega}}^{\boldsymbol{R}}$ is the vector $\boldsymbol{n_\omega^R}$ scaled on the plane identified by the vectors $\boldsymbol{c_i}$ and parallel to the Listing plane. At this point we can define the vector:

$$\hat{\boldsymbol{n}}_{\boldsymbol{\omega}}^{\boldsymbol{R}} = 2\tilde{\boldsymbol{n}}_{\boldsymbol{\omega}}^{\boldsymbol{R}} = [\hat{n}_{\omega x}^R \; \hat{n}_{\omega y}^R \; \hat{n}_{\omega z}^R]^T \tag{35}$$

We can chose two vectors like: $\hat{\boldsymbol{n}}_{\boldsymbol{H}}^{\boldsymbol{\omega}} = [\hat{n}_{\omega x}^R \; 0 \; \tilde{n}_{\omega z}^R]^T$ and $\hat{\boldsymbol{n}}_{\boldsymbol{V}}^{\boldsymbol{\omega}} = [0 \; \hat{n}_{\omega y}^R \; \tilde{n}_{\omega z}^R]^T$ that:

$$\hat{\boldsymbol{n}}_{\boldsymbol{\omega}}^{\boldsymbol{R}} = \hat{\boldsymbol{n}}_{\boldsymbol{H}}^{\boldsymbol{\omega}} + \hat{\boldsymbol{n}}_{\boldsymbol{V}}^{\boldsymbol{\omega}} \qquad \text{and} \qquad \tilde{\boldsymbol{n}}_{\boldsymbol{\omega}}^{\boldsymbol{R}} = \frac{\hat{\boldsymbol{n}}_{\boldsymbol{H}}^{\boldsymbol{\omega}} + \hat{\boldsymbol{n}}_{\boldsymbol{V}}^{\boldsymbol{\omega}}}{2} \tag{36}$$

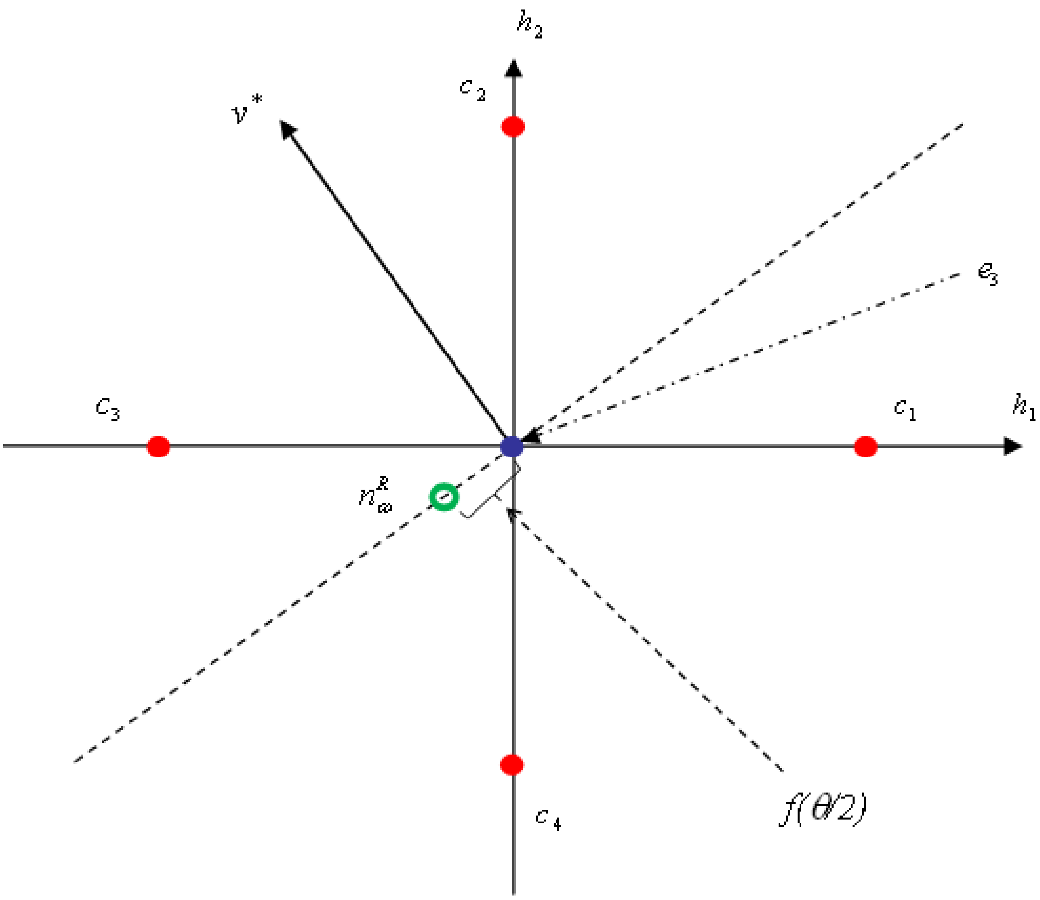


Figure 10: Front view of the geometry of the oculomor plant.

An important property of the vectors $\hat{\boldsymbol{n}}_{\boldsymbol{H}}^{\boldsymbol{\omega}}$ and $\hat{\boldsymbol{n}}_{\boldsymbol{V}}^{\boldsymbol{\omega}}$ is that they belong, respectively, to the planes identified by vectors $\boldsymbol{c_1}$,$\boldsymbol{c_3}$ and $\boldsymbol{c_2}$,$\boldsymbol{c_4}$. These two vectors can be written as the linear combination of vectors $\boldsymbol{c_1}$,$\boldsymbol{c_3}$ and $\boldsymbol{c_2}$,$\boldsymbol{c_4}$:

$$\begin{aligned} \hat{\boldsymbol{n}}_{\boldsymbol{H}}^{\boldsymbol{\omega}} &= \gamma_1^o \boldsymbol{c_1} + \gamma_3^o \boldsymbol{c_3} \\ \hat{\boldsymbol{n}}_{\boldsymbol{V}}^{\boldsymbol{\omega}} &= \gamma_2^o \boldsymbol{c_2} + \gamma_4^o \boldsymbol{c_4} \end{aligned} \tag{37}$$

The four unknown variables γ_i^p are computed as:

$$\begin{aligned}
\gamma_1^o &= \frac{\hat{\boldsymbol{n}}_{\boldsymbol{H}}^{\boldsymbol{\omega}} \cdot \boldsymbol{c_1} - (\boldsymbol{c_3} \cdot \boldsymbol{c_1})(\hat{\boldsymbol{n}}_{\boldsymbol{H}}^{\boldsymbol{\omega}} \cdot \boldsymbol{c_3})}{1 - (\boldsymbol{c_3} \cdot \boldsymbol{c_1})^2} \\
\gamma_3^o &= \frac{\hat{\boldsymbol{n}}_{\boldsymbol{H}}^{\boldsymbol{\omega}} \cdot \boldsymbol{c_3} - (\boldsymbol{c_3} \cdot \boldsymbol{c_1})(\hat{\boldsymbol{n}}_{\boldsymbol{H}}^{\boldsymbol{\omega}} \cdot \boldsymbol{c_1})}{1 - (\boldsymbol{c_3} \cdot \boldsymbol{c_1})^2} \\
\gamma_2^o &= \frac{\hat{\boldsymbol{n}}_{\boldsymbol{H}}^{\boldsymbol{\omega}} \cdot \boldsymbol{c_2} - (\boldsymbol{c_4} \cdot \boldsymbol{c_2})(\hat{\boldsymbol{n}}_{\boldsymbol{H}}^{\boldsymbol{\omega}} \cdot \boldsymbol{c_4})}{1 - (\boldsymbol{c_4} \cdot \boldsymbol{c_2})^2} \\
\gamma_4^o &= \frac{\hat{\boldsymbol{n}}_{\boldsymbol{H}}^{\boldsymbol{\omega}} \cdot \boldsymbol{c_4} - (\boldsymbol{c_4} \cdot \boldsymbol{c_2})(\hat{\boldsymbol{n}}_{\boldsymbol{H}}^{\boldsymbol{\omega}} \cdot \boldsymbol{c_2})}{1 - (\boldsymbol{c_4} \cdot \boldsymbol{c_2})^2}
\end{aligned} \tag{38}$$

At this point the muscle forces related to the general solution of the homogeneous equation are:

$$f_i^o = \frac{\gamma_i^o |\hat{\boldsymbol{n}}_{\boldsymbol{H}}^{\boldsymbol{\omega}} \times \boldsymbol{c_i}|}{r} \qquad \forall i = 1 \dots 4 \tag{39}$$

and the total force f_i produced by each rectus muscle is:

$$f_i = f_i^o + f_i^p \qquad \forall i = 1 \dots 4 \tag{40}$$

5 Simulations

In order to verify the validity of the solution for the static direct problem and for the static inverse problem have been developed two different MATLAB programs. In Appendix A is shown the code to compute the solution for the direct and inverse problems. With the *inverse_ problem* function have beeen computed the muscle forces at different eye orientaion. The angular position of the eye is defined by the rotaion axis $\boldsymbol{v}$ and the amount of rotation θ about $\boldsymbol{v}$, defined as:

$$(\boldsymbol{v}^*, \theta^*) = ([\cos\alpha \ \sin\alpha \ 0], \theta^*) \tag{41}$$

In Figure 11 are shown the module of the muscls forces at different eye orientation. The four muscle forces computed at different eye orientation has been used

α [deg]	θ [deg]	f1 [N]	f2 [N]	f3 [N]	f4 [N]
0	20	0.7344	0.6729	0.7344	0.7506
30	20	0.7471	0.6846	0.7106	0.7505
45	20	0.7496	0.6969	0.6969	0.7496
60	20	0.7505	0.7106	0.6846	0.7471
90	20	0.7506	0.7344	0.6729	0.7344

Figure 11: Table with the computed muscle forces given a reference eye orientation. f_1,f_2,f_3 and f_4 are, respectively the forces of the lateral, superior, medial and inferior rectus muscle, expressed in Newton

like inputs for the Simulink model of the oculomotor plant shown in Figure 12.
The human eye system is composed of the model of the head (supposed fixed) and the eyeball (Simulink Library blocks), modeled on the assumptions made in the Section 3. This simulator is described in detail in the Deliverable 1.4b entitled *Bioinspired Stereovision Robot System. Robot Prototype Simulator*.
In this simulations we test the algorithms on one eye and we assume that the eye start always from the primary position.
Each block of the system has a custom *Graphical User Interface* (MATLAB GUI) where you can configure the geometrical and mechanical parameters of the block. On the head model you can configure the initial position and orientation of the head (the head is supposed fixed) and the initial orientation and position of the eyeball with respect to the head reference system. On the eye block are configured the following parameters (used also to solve the direct and inverse static problem) [22]:

- Mechanical parameters
  - Orbit elasticity Kp [Nm^{-1}]: 66.4,
  - Orbit viscosity Bp [Nsm^{-1}]: 3.1,

- Inertia moment Jp $[Ns^2m^{-1}]$: 2.2x10^{-3},
- Mass [g]: 8.

- Geometrical parameters:
  - Radius r [mm]: 12,
  - Insertion points $\boldsymbol{c_i}$ [mm]: computed as explained is Section 3,
  - Pointwise pulleys $\boldsymbol{p_i}$ [mm]: computed as explained is Section 3.

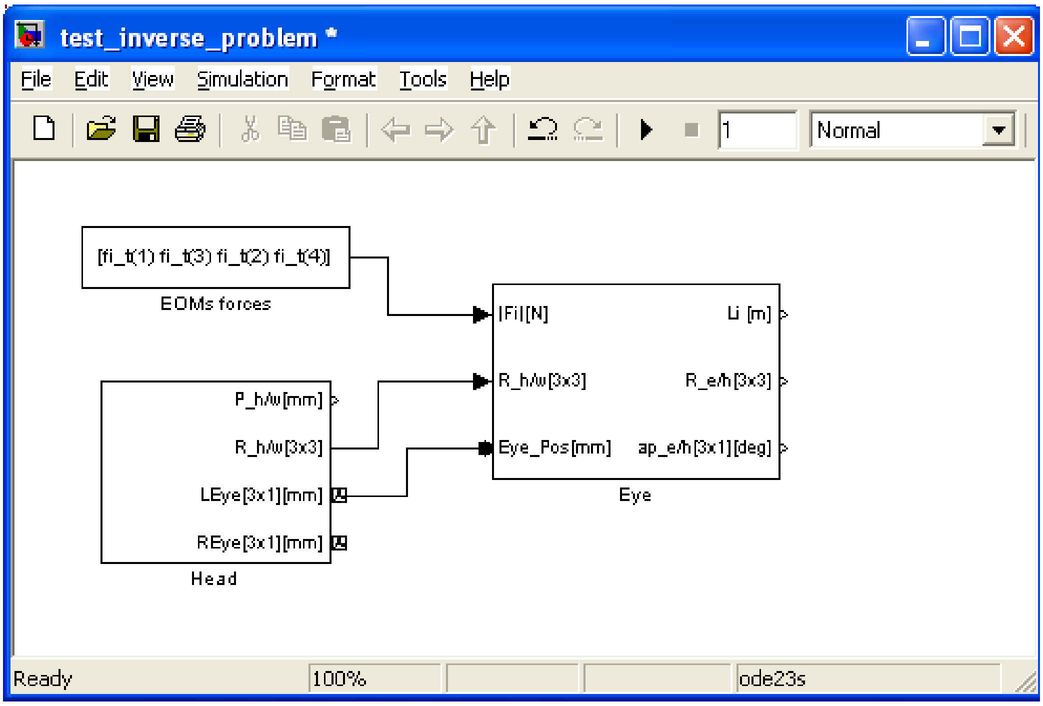


Figure 12: Simulink model of the oculomotor plant, composed of the head, the eyeball and the four muscle forces (constats).

In Figure 13 is shown the output (angular position of the eye) of the Simulink model of the oculomotor plant where the inputs are the forces computed from the reference eye orientation:

$$(\boldsymbol{v}^*, \theta^*) = ([\cos\alpha \;\; \sin\alpha \;\; 0], \theta^*) \qquad \text{where} \qquad \alpha = 30^\circ \text{ and } \theta = 20^\circ$$

As it appears in Figure 13 the third component of the eye orientation vector is equal to zero, according to the Listing's Law.
In order to verify the validity of the static direct problem algorithm, four muscle

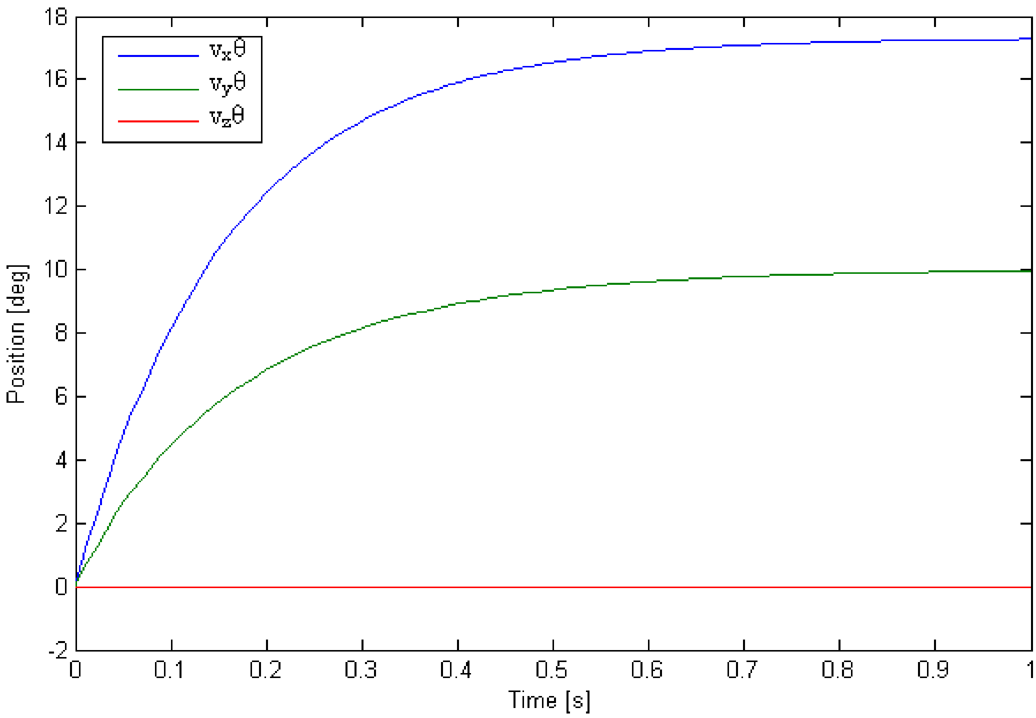


Figure 13: Eye orientation after the action of the EOMs. $v_x\theta$, $v_y\theta$ $v_z\theta$ components of the eye orientation vector.

f = [0.5635, 0.4786, 0.5026, 0.5867]

Algorithm steps:

1 $\theta = 9.7299°$
$\mathbf{v} = [0.8711 \quad 0.4912 \quad 0]$

2 $\theta = 14.6904°$
$\mathbf{v} = [0.8710 \quad 0.4912 \quad 0]$

3 $\theta = 17.2352°$
$\mathbf{v} = [0.8703 \quad 0.4926 \quad 0]$

.
.
.

20 $\theta = 20.0000°$
$\mathbf{v} = [0.8660 \quad 0.5000 \quad 0]$

Figure 14: Static Direct Problem algorithm. The muscle forces (inputs of the algorithm), and the orientation vector (output of the algotithm) are shown

forces have been passed as inputs to the *direct_problem* function and in Figure 14 are shown the muscle forces, the steps of the algorithm and the solution.

6 Conclusions

In this document a detailed model, geometrical and mechanical, of the human oculomotor plant has been described. In particular the mechanical properties of the orbital tissue that surround the eyeball have been analyzed. Furthermore the geometrical parameters (extraocular muscle insertion points, softwise pulleys) which are fundamental for the implementation of the ocular laws (Listing's law and Half-Angle Rule) have been presented.

Two different models of the extraocular muscle have been described: the first is a non-linear model based on the Hill modelization of the skeletal muscle, the second is a linearized model composed of linear mechanical elements such as spring and dumper.

The main problem of artificial and biological systems is to compute the forces that drive the system in a reference position (inverse dynamic problem) and to compute the position given the forces (direct problem). In this document a recoursive algorithm able to solve the direct problem in few steps is described. Furthermore a geometrcial inverse control problem is formulated and numerically solved. These two problems are solved with the system at steady state, namely when the system is in a static condition.

On this model the analysis of the dynamic behavior of the system can be computed and the control problem for the saccadic and smooth pursuit movements can be formulated.

Both the oculomotor model and the control strategy, presented here, can be easily integrated with the vision module developed in the other workpackages (WP2, WP3), infact given a retinal position (of the target object) are computed the muscle forces that drive the eyeball in that position.

Furthermore the robot eye prototype is developed on the assumpions given in this document, regarding the eyeball and the actuation system (linear motor implementing the extra-ocular muscle) that drive the eyeball respecting the two ocular laws. The control strategy can be also implemented on the robot with a force control on the motor, instead themore classical position or velocity control strategy.

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A Simulation Code

A.1 Inverse Problem (MATLAB Code)

```
function inverse_problem(alfa,teta)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%The INVERSE_PROBLEM(ALFA,TETA) function computes the four muscle forces
%%from a given eye orientation in terms of alfa and teta.
%%alfa [deg] identify the orientaion vector v
%%teta [deg] is the amount of rotation about v
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% problem initializtion
s = init_inverse_problem(alfa,teta);

% computing the particular solution
lp = (s.Kp*rad2deg(s.teta)/(4*norm(s.nw_r)*cos(s.beta)*(sin(s.teta2))));
% computing the general solution related to the homogeneous equation
lo = force(s);

l = lo+lp;
% total muscle force
f1_t = l(1)*(norm(cross(s.nw_r,s.c1)))/s.r;
f2_t = l(2)*(norm(cross(s.nw_r,s.c2)))/s.r;
f3_t = l(3)*(norm(cross(s.nw_r,s.c3)))/s.r;
f4_t = l(4)*(norm(cross(s.nw_r,s.c4)))/s.r;
% muscle force related to lp
f1_p = lp*(norm(cross(s.nw_r,s.c1)))/s.r;
f2_p = lp*(norm(cross(s.nw_r,s.c2)))/s.r;
f3_p = lp*(norm(cross(s.nw_r,s.c3)))/s.r;
f4_p = lp*(norm(cross(s.nw_r,s.c4)))/s.r;
% muscle force related to lo
f1_o = lo(1)*(norm(cross(s.nw_r,s.c1)))/s.r;
f2_o = lo(2)*(norm(cross(s.nw_r,s.c2)))/s.r;
f3_o = lo(3)*(norm(cross(s.nw_r,s.c3)))/s.r;
f4_o = lo(4)*(norm(cross(s.nw_r,s.c4)))/s.r;

% store the forces in a MAT file
fi_t =[f1_t f2_t f3_t f4_t];
fi_p =[f1_p f2_p f3_p f4_p];
fi_o =[f1_o f2_o f3_o f4_o];
a = num2str(alfa);
t = num2str(teta);
save(['force_' flag, '_' a '_' t, '.mat'],'fi_t','fi_p','fi_o','lo','lp');

function s = init_inverse_problem(alfa,teta)
    r =11*0.001;                         % radius
    s.r = r;
```

```
    s.Kp = 66.4*(r)^2/57.296;              % orbit elasticity
    h3 = [0 0 1]';
    s.alfa = deg2rad(alfa);
    s.v = [cos(s.alfa) sin(s.alfa) 0]'; % orientation vector
    s.teta = deg2rad(teta);                % amount of rotation
    s.beta = deg2rad(55);                  % angle for pulleys and inserion points
    s.teta2 = s.teta/2;

    delta = cos(s.beta)/cos(s.teta2);    % scale factor

    % compute rotation matrix
    S=[0 -s.v(3) s.v(2); s.v(3) 0 -s.v(1);-s.v(2) s.v(1) 0];
    R = eye(3)+S*sin(s.teta)+S^2*(1-cos(s.teta));

    % insertion points in primary position
    C1 = r * [sin(55*pi/180) ; 0.00 ; cos(55*pi/180)];
    C3 = r * [-sin(55*pi/180) ; 0.00 ; cos(55*pi/180)];
    C2 = r * [0.00 ; sin(55*pi/180) ; cos(55*pi/180)];
    C4 = r * [0.00; -sin(55*pi/180) ; cos(55*pi/180)];
    s.c1 = C1/norm(C1);
    s.c2 = C2/norm(C2);
    s.c3 = C3/norm(C3);
    s.c4 = C4/norm(C4);

    % nw and nw_r and nw_r1(scaled)
    nw = (h3*cos(s.teta2)-(cross(h3,s.v)*sin(s.teta2)));
    s.nw_r = R'*nw;
    s.nw_r1 = delta * s.nw_r;

function loi = force(s)

    nw_r2 = 2*s.nw_r1;
    % nh and nv vectors
    nh = [nw_r2(1) 0 s.nw_r1(3)]'
    nv = [0 nw_r2(2) s.nw_r1(3)]';
    % compute general solution
    lo1 = ((nh'*s.c1)-((s.c3'*s.c1)*(nh'*s.c3)))/(1-(s.c1'*s.c3)^2)
    lo3 = ((nh'*s.c3)-((s.c3'*s.c1)*(nh'*s.c1)))/(1-(s.c1'*s.c3)^2)
    lo2 = ((nv'*s.c2)-((s.c4'*s.c2)*(nv'*s.c4)))/(1-(s.c2'*s.c4)^2);
    lo4 = ((nv'*s.c4)-((s.c4'*s.c2)*(nv'*s.c2)))/(1-(s.c2'*s.c4)^2);

    loi = [lo1 lo2 lo3 lo4]*s.r;
```

A.2 Direct Problem (MATLAB Code)

```
function [v,teta]=direct_problem(fi)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%The [V,TETA]=DIRECT_PROBLEM(Fi) function computes the eye
%%orientation given the four muscle forces.
```

```
%%Fi vector with the four muscle forces
%%V identify the orientaion vector v
%%TETA [deg] is the amount of rotation about v
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc
r =11*0.001;                    % eyeball radius
Kp = 66.4*(r)^2/57.296;         % orbit elasticity
h3 = [0 0 1]';

global f  c1 c2 c3 c4 i
i=1
f = fi;
% insertion points in primary position
C1 = r * [sin(55*pi/180) ; 0.00 ; cos(55*pi/180)];
C3 = r * [-sin(55*pi/180) ; 0.00 ; cos(55*pi/180)];
C2 = r * [0.00 ; sin(55*pi/180) ; cos(55*pi/180)];
C4 = r * [0.00; -sin(55*pi/180) ; cos(55*pi/180)];
c1 = C1/norm(C1);
c2 = C2/norm(C2);
c3 = C3/norm(C3);
c4 = C4/norm(C4);

nw_r = h3;                  % initialization of nw_r for the recoursive alg
[v teta] = alg_(nw_r,h3,Kp,r);


function [w_v teta] = alg_(nw_r,h3,Kp,r)
    global f  c1 c2 c3 c4 i
    D = [1/(norm(cross(nw_r,c1))) 0 0 0; 0 1/(norm(cross(nw_r,c2))) 0 0;...
        0 0 1/(norm(cross(nw_r,c3))) 0; 0 0 0 1/(norm(cross(nw_r,c4)))];
    l = D*f'*r;
    h = l(1)*c1 + l(2)*c2+l(3)*c3+l(4)*c4;
    w = cross(nw_r,h);
    w_v = w/norm(w);
    teta = norm(w)/Kp;
    temp1 =teta;
    temp = nw_r;
    nw_r = (h3*cos(deg2rad(temp1/2))+(cross(h3,w_v)*sin(deg2rad(temp1/2))));
    if (temp-nw_r<1e-12)

    else
        [w_v teta] = alg_(nw_r,h3,Kp,r)
        i = i+1
    end
```